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Discrete Mathematics 266 (2003) 311–320
DISCRETE MATHEMATICS
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Every cubic cage is quasi 4-connected

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Received 4 July 2001; received in revised form 13 February 2002; accepted 12 August 2002

Abstract

A (δ, g)-*cage* is a regular graph of degree δ and girth g with the least possible number of vertices. It was proved by Fu, Huang and Rodger that every $(3, g)$-cage is 3-connected. Moreover, the same authors conjectured that all (δ, g)-cages are δ-connected for every $\delta \geqslant 3$. As a first step towards the proof of this conjecture, Jiang and Mubayi, and independently Daven and Rodger, showed that every (δ, g)-cage with $\delta \geqslant 3$ is 3-connected. A 3-connected graph G is called *quasi* 4-*connected* if for each cutset $S \subset V(G)$ with $|S| = 3$, S is the neighbourhood of a vertex of degree 3 and $G - S$ has precisely two components. In this paper, we prove that every $(3, g)$-cage with $g \geqslant 5$ is quasi 4-connected, which can be seen as a further step towards the proof of the aforementioned *conjecture*.

MSC: 05C35; 05C40

Keywords: Quasi 4-connected graph; Cage; Connectivity; Superconnectivity; Cutset

1. Introduction

Throughout this paper, all the graphs are *simple*, that is, without loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every $v \in V$, $N(v)$ denotes the *neighbourhood* of v, that is, the set of all vertices adjacent to v. If $S \subset V$, then $N(S) = \bigcup_{v \in S} N(v)$. If H is a subgraph of G, then $N_H(S) = N(S) \cap V(H)$. The subgraph of G induced by S is denoted $G[S]$. For $u, v \in V$, $d(u, v) = d_G(u, v)$ denotes the *distance* between u and v, that is, the length of a shortest (u, v)-path. Similarly, for $v \in V$ and $S \subset V$, $d(v, S) = \min\{d(v, s) : s \in S\}$. The *diameter*

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0012-365X/03/$ - see front matter
doi:10.1016/S0012-365X(02)00816-6

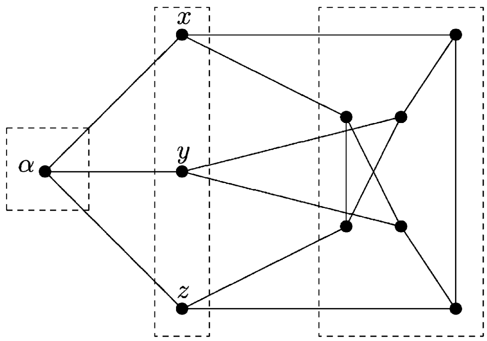

Fig. 1. The Petersen graph is quasi 4-connected.

$D = D(G)$ is the maximum distance over all pairs of vertices in G. A graph G is called *connected* if every pair of vertices is joined by a path, that is, if $D < \infty$. If $S \subset V$ and $G - S$ is not connected, then S is said to be a *cutset* or a *separating set*. Certainly, every connected graph different from a complete graph has a cutset. A (*connected*) *component* of a nonconnected graph G is a maximal connected subgraph of G. A connected graph is called *k-connected* if every cutset has cardinality at least k. The *connectivity* κ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *minimum* cutsets are those having cardinality κ. A minimum cutset S is called *trivial* if $S = N(v)$ for some $v \in V \setminus S$.

The degree of a vertex v is $\deg(v) = |N(v)|$, whereas the (minimum) degree δ of G is the minimum degree over all vertices of G. A graph is called *regular* if all its vertices have the same degree. The degree of a vertex v in a subgraph H of G is $\deg_H(v) = |N(v) \cap V(H)| = |N_H(v)|$. The *girth* $g = g(G)$ is the length of a shortest cycle in G. A (δ, g)-*graph* is a regular graph of degree δ and girth g. A graph G of minimum degree δ is called *superconnected* if $\kappa = \delta$ and every minimum cutset is trivial (see [5]). A 3-connected $(3, g)$-graph is *quasi* 4-*connected* if for each cutset $S \subset V(G)$ with $|S| = 3$, S is the neighbourhood of a vertex of degree 3 (i.e., G is superconnected), and $G - S$ has exactly two components (see Fig. 1). Quasi 4-connected graphs, which exhibit many of the properties of 4-connected graphs, offer a true refinement of the strict vertex connectivity (see [1] for more details).

Let $f(\delta, g)$ denote the smallest integer v such that there exists a (δ, g)-graph having v vertices. A (δ, g)-*cage* is a (δ, g)-graph with $f(\delta, g)$ vertices. These graphs have been intensely studied since introduced by Tutte [9] (see [10] for a survey, see also [8]). Most of the work carried out so far has focused on the existence problem, whereas very little is known about structural properties. Recently, several authors have approached the problem of studying the connectivity of cages (see [3,6,7]). In the first paper on this issue (see [6]), Fu et al. proved that every (δ, g)-cage is 2-connected. In addition, they conjectured that all (δ, g)-cages are δ-connected and proved this statement for $\delta = 3$. Subsequently, it has been proved that every (δ, g)-cage with $\delta \geqslant 3$ is 3-connected (see [3,7]).

This paper puts forward a further contribution towards the proof of the mentioned *conjecture* showing that every $(3,g)$-cage with $g \geqslant 5$ is quasi 4-connected. This statement has been proved taking into account the following known results.

Theorem 1.1 (Fàbrega and Fiol [5]). *Let G be a connected graph with minimum degree $\delta \geqslant 3$ and diameter D. Then, G is superconnected if $D \leqslant 2\lfloor (g-3)/2 \rfloor$.*

Theorem 1.2 (Erdös and Sachs [4], Fu et al. [6]). *If $\delta \geqslant 3$ and $3 \leqslant g_1 < g_2$, then $f(\delta, g_1) < f(\delta, g_2)$.*

Theorem 1.3 (Jiang and Mubayi [7]). *Let S be a cutset of a (δ, g)-cage with $\delta \geqslant 3$ and $g \geqslant 5$. Then, the diameter of $G[S]$ is at least $\lfloor g/2 \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u,v)$ is maximized for exactly one pair of vertices.*

The only $(3,g)$-cages with $g=3$ and 4 are K_4 and $K_{3,3}$, respectively. Certainly, the complete graph K_4 has no cutsets. It is also clear that the complete bipartite graph $K_{3,3}$ is superconnected but not quasi 4-connected. For this reason, we henceforth assume that $g \geqslant 5$.

2. Every cubic cage is quasi 4-connected

It is well known that the Petersen graph P is the unique $(3,5)$-cage. Since P has diameter 2, we conclude that P is superconnected as a consequence of Theorem 1.1. It is also easy to see that it is quasi 4-connected (see Fig. 1). Let G be a $(3,g)$-cage with $g \geqslant 6$. Consider the set $\mathscr{F}$ of all nontrivial minimum (i.e., of cardinality 3) cutsets of G. The first goal of this section is to show that $\mathscr{F} = \emptyset$, i.e., that G is superconnected. To this end, suppose on the contrary that $\mathscr{F} \neq \emptyset$. For every $F \in \mathscr{F}$, let C_F denote a smallest component of $G-F$, that is, such that $|V(C_F)| \leqslant |V(C)|$ for every component C of $G-F$. Notice that $|V(C_F)| \geqslant 2$, since F is nontrivial. Let us show that $|V(C_F)| \geqslant g$, which is clear if C_F contains a cycle. If C_F is acyclic, there must exist two vertices $v, w \in V(C_F)$ with $|N_{C_F}(v)| = |N_{C_F}(w)| = 1$, and thus, there exists a vertex $f \in F$ such that $f \in N(v) \cap N(w)$. This implies that the shortest (v,w)-path in C_F has length at least $g-2$, that is, $|V(C_F)| \geqslant g-1$. But if $|V(C_F)| = g-1$, then C_F is a path of length $g-2 \geqslant 4$ and thus, there exist in C_F two vertices different from v and w whose sets of neighbours intersect in F since $|F| = 3$, contradicting the fact that the girth of G is equal to $g \geqslant 6$.

Let $S = \{x, y, z\}$ denote any nontrivial minimum cutset of G satisfying:

$$|V(C_S)| \leqslant |V(C_F)|, \quad \text{for every } F \in \mathscr{F}. \tag{1}$$

In the rest of this work, we use the following notation: $C_1 = C_S$, $C_2 = (G-S) \setminus C_1$, $X = N_{C_1}(x)$, $Y = N_{C_1}(y)$, and $Z = N_{C_1}(z)$.

Lemma 2.1. *If S is a nontrivial minimum cutset satisfying (1), then $|X| = |Y| = |Z| = 2$, C_2 is connected and $2|V(C_1)| \leqslant |V(G)| - 6$.*

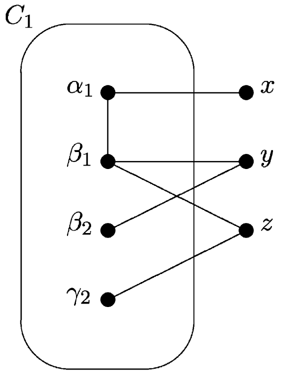

Fig. 2. Detail of a nonsuperconnected $(3,g)$-cage when $X=\{\alpha_1\}$.

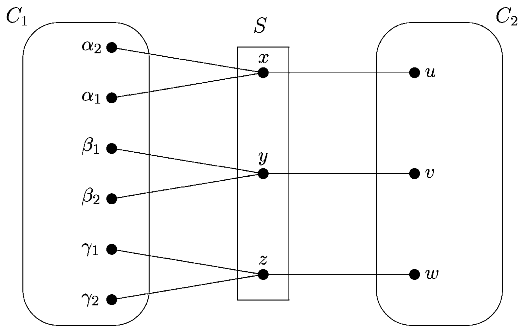

Fig. 3. Structure of a nonsuperconnected $(3,g)$-cage with $g \geqslant 6$.

Proof. Suppose $X=\{\alpha_1\}$. Consider the set $F=\{\alpha_1,y,z\}$, which is clearly a minimum cutset satisfying $|V(C_F)|<|V(C_1)|$. In consequence, it must be trivial. Let β_1 be a vertex such that $F=N(\beta_1)$. Observe that $\beta_1 \neq x$, since otherwise the diameter of the subgraph $G[S]$ is equal to 2, contradicting Theorem 1.3 because $g \geqslant 6$. As a consequence, we can conclude that $\beta_1 \in V(C_1)$. Suppose $N_{C_1}(y)=\{\beta_1\}$ [resp. $N_{C_1}(z)=\{\beta_1\}$] and consider the set $F'=\{\alpha_1,\beta_1,z\}$ [resp. $F'=\{\alpha_1,\beta_1,y\}$]. Observe that F' is a minimum cutset whose induced subgraph $G[F']$ has diameter 2, which contradicts Theorem 1.3. We can thus assume that $|N_{C_1}(y)|=|N_{C_1}(z)|=2$, i.e., $N_{C_1}(y)=\{\beta_1,\beta_2\}$, $N_{C_1}(z)=\{\beta_1,\gamma_2\}$ (see Fig. 2), with $\beta_2 \neq \gamma_2$ since $g \geqslant 6$. Finally, take the set $F''=\{\alpha_1,\beta_2,\gamma_2\}$, which is a minimum cutset since $N(\beta_1)=\{\alpha_1,y,z\}$ and $|V(C_1)| \geqslant g \geqslant 6$. Moreover, F'' is nontrivial because $d_{C_1}(\alpha_1,\beta_2) \geqslant g-3 \geqslant 3$, since $\alpha_1\beta_1 y\beta_2$ is a path in G of length three. As $|V(C_{F''})|<|V(C_1)|$, we arrive again at a contradiction. So, we have proved that $|X|=|Y|=|Z|=2$. Observe that from this fact we obtain that $|N_{C_2}(S)|=3$, from which we derive that C_2 must be connected, since S is a minimum cutset and $\delta=3$, whence the only possible structure is that displayed in Fig. 3. Besides, as G is a cubic graph, $|V(G)|$ must be even, which implies that $|V(C_2)|=|V(C_1)|+v$, where $v \geqslant 1$ is an odd number. Suppose that $|V(C_2)|=|V(C_1)|+1$, and consider $N_{C_2}(x)=\{u\}$. Then, the set $H=\{u,y,z\}$ is a minimum nontrivial cutset satisfying $|V(C_H)|=|V(C_1)|$ (if H was trivial, then $N_{C_2}(y)=N_{C_2}(z)=\{v\}$, and $\{x,v\}$ would be a cutset of G). Therefore, by the above proved, $|N_{C_H}(y)|=|N_{C_H}(z)|=2$, which is a contradiction because $|N_{C_H}(y)|=|N_{C_2}(y)|=1$, and similarly for vertex z. Hence $|V(C_2)| \geqslant |V(C_1)|+3$, that is, $2|V(C_1)|+6 \leqslant |V(G)|$. □

From this result, we can conclude that for every nontrivial minimum cutset $S=\{x,y,z\}$ satisfying (1), the only possible structure is that displayed in Fig. 3. That is, $V(G)=V(C_1)\cup S\cup V(C_2)$; and $N_{C_1}(x)=\{\alpha_1,\alpha_2\}=X$, $N_{C_1}(y)=\{\beta_1,\beta_2\}=Y$ and $N_{C_1}(z)=\{\gamma_1,\gamma_2\}=Z$. In this context, we introduce L as

$$L=\min\{d_{C_1}(X,Y),d_{C_1}(X,Z),d_{C_1}(Y,Z)\}$$

Under the assumption that G is a nonsuperconnected $(3,g)$-cage with $g \geqslant 6$, for which a certain nontrivial minimum cutset $S = \{x, y, z\}$ satisfying (1) is arbitrarily chosen, we put forward two lemmas providing both a lower and an upper bound of L.

Lemma 2.2. *If G is a nonsuperconnected $(3,g)$-cage with $g \geqslant 8$, then $L \geqslant 1$.*

Proof. Assume that G is a nonsuperconnected $(3,g)$-cage with $g \geqslant 8$ and $L = 0$. Without loss of generality, suppose that $d_{C_1}(X,Y) = 0$ and $\alpha_1 = \beta_1$, which implies that $d_{C_1}(\alpha_1, \beta_2) \geqslant g-2$ and $d_{C_1}(\alpha_2, \beta_2) \geqslant g-4$. (See Fig. 4).

Notice that $\alpha_1 \neq \gamma_i$, $i = 1,2$, because the cutset $S = \{x, y, z\}$ is nontrivial. Note also that any (β_2, γ_i)-path in C_1 cannot contain the vertex α_1, since $\deg_{C_1}(\alpha_1) = 1$. Hence, considering a shortest (β_2, γ_i)-path in C_1 we obtain that

$$x\alpha_1 y \beta_2 \cdots \gamma_i z$$

is a separating path P of G, whose length is $4 + d_{C_1}(\beta_2, \gamma_i)$. Applying Theorem 1.3 to $V(P)$ and taking into account that $g \geqslant 8$, it follows that $4 + d_{C_1}(\beta_2, \gamma_i) > \lfloor g/2 \rfloor \geqslant 4$, hence $d_{C_1}(\beta_2, \gamma_i) \geqslant 1$. Similarly, it is proved that $d_{C_1}(\alpha_2, \gamma_i) \geqslant 1$. Next, consider the set $N_{C_1}(\alpha_1) = \{u\}$, which clearly satisfies $u \notin \{\alpha_2, \beta_2\}$. Furthermore, $u \notin \{\gamma_1, \gamma_2\}$, because otherwise the subgraph induced by the cutset $\{x, \alpha_1, u, z, y\}$ would have diameter 3, contradicting Theorem 1.3. As a consequence, we have proved that the set $\Omega = \{u, \alpha_2, \beta_2, \gamma_1, \gamma_2\}$ has cardinality five (see Fig. 4). Consider the subgraph $\tilde{C}_1 = C_1 - \{\alpha_1\}$, which is clearly connected since $\deg_{C_1}(\alpha_1) = 1$. Notice that for all $v \in V(\tilde{C}_1) \setminus \Omega$, $\deg_{\tilde{C}_1}(v) = 3$, and for all $w \in \Omega$, $\deg_{\tilde{C}_1}(w) = 2$. Observe also that:

- $d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g-2$, since $d_G(\gamma_1, \gamma_2) = 2$.
- $d_{\tilde{C}_1}(w_1, w_2) \geqslant g-4$, for every two different vertices $w_1, w_2 \in \Omega \setminus \{\gamma_1, \gamma_2\}$, as $d_G(w_1, w_2) \leqslant 4$.
- $d_{\tilde{C}_1}(w_1, \gamma_1) + d_{\tilde{C}_1}(w_2, \gamma_2) \geqslant g-6$, for any pair of different vertices $w_1, w_2 \in \Omega \setminus \{\gamma_1, \gamma_2\}$, since the union of any (w_1, γ_1)-path in $\tilde{C}_1$, any (w_2, γ_2)-path in $\tilde{C}_1$, the (w_1, w_2)-path in G of length at most four, and the (γ_1, γ_2)-path in G of length two, contains a cycle in G.

We now construct a new graph G^* as follows. Let $\tilde{C}'_1$ be a copy of $\tilde{C}_1$ such that $V(\tilde{C}_1) \cap V(\tilde{C}'_1) = \emptyset$, and let φ be a bijection between $\tilde{C}_1$ and $\tilde{C}'_1$ such that $\varphi(\gamma_1) = \gamma'_2$, $\varphi(\gamma_2) = \gamma'_1$, and for all $v \in V(\tilde{C}_1) \setminus \{\gamma_1, \gamma_2\}$, $\varphi(v) = v'$. Let G^* be the graph such that $V(G^*) = V(\tilde{C}_1) \cup V(\tilde{C}'_1)$, and $E(G^*) = E(\tilde{C}_1) \cup E(\tilde{C}'_1) \cup E^+$, where $E^+ = \{w\varphi(w) : w \in \Omega\} = \{uu', \alpha_2\alpha'_2, \beta_2\beta'_2, \gamma_1\gamma'_2, \gamma_2\gamma'_1\}$. Note that G^* is a 3-regular connected graph satisfying $|V(G^*)| < |V(G)|$. (See Fig. 5). Hence, it suffices to show that $g(G^*) \geqslant g(G) = g$ to get a contradiction with the fact that G is a $(3,g)$-cage (see Theorem 1.2). To this end, let $\mathscr{C}$ be a cycle of G^*. Since E^+ is an edge cut, $\mathscr{C}$ contains an even number of edges from E^+. If $E(\mathscr{C}) \cap E^+ = \emptyset$, then $\mathscr{C}$ corresponds to a cycle in G, and its length is at least g.

Suppose $|E(\mathscr{C}) \cap E^+| = 2$. Assume first that $\{\gamma_1\gamma'_2, \gamma_2\gamma'_1\} \subset E(\mathscr{C})$. Since $d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g-2$, $\mathscr{C}$ has length at least $g-2+2 = g$. Secondly, suppose that $\{\gamma_1\gamma'_2, ww'\} \subset E(\mathscr{C})$,

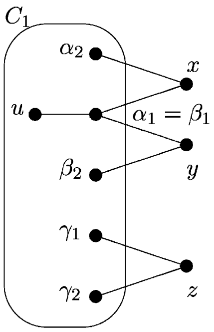

Fig. 4. Detail of a nonsuperconnected $(3, g)$-cage with $L = 0$.

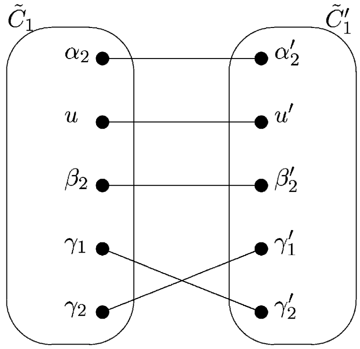

Fig. 5. A new graph G^*.

where $w \in \Omega \setminus \{\gamma_1, \gamma_2\}$. By the triangular inequality, we get $d_{\tilde{C}_1}(w, \gamma_1) + d_{\tilde{C}_1'}(w', \gamma_2') = d_{\tilde{C}_1}(w, \gamma_1) + d_{\tilde{C}_1}(w, \gamma_2) \geqslant d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g - 2$, whence we obtain that the length of $\mathscr{C}$ is at least $g - 2 + 2 = g$. The same reasoning is applied if $\{\gamma_2\gamma_1', ww'\} \subset E(\mathscr{C})$, $w \in \Omega \setminus \{\gamma_1, \gamma_2\}$. Finally, suppose that $\{w_1w_1', w_2w_2'\} \subset E(\mathscr{C})$, with $w_1, w_2 \in \Omega \setminus \{\gamma_1, \gamma_2\}$. As $d_{\tilde{C}_1}(w_1, w_2) + d_{\tilde{C}_1'}(w_1', w_2') = 2d_{\tilde{C}_1}(w_1, w_2) \geqslant 2(g - 4)$, we can assure that $\mathscr{C}$ has length at least $2(g - 4) + 2 > g$, since $g \geqslant 8$.

Suppose $|E(\mathscr{C}) \cap E^+| = 4$. Assume first that $\{\alpha_2\alpha_2', uu', \beta_2\beta_2'\} = \{w_1w_1', w_2w_2', w_3w_3'\} \subset E(\mathscr{C})$. Then, $\mathscr{C}$ must contain an (w_i, w_j)-path and a (w_j', w_r')-path, $\{i, j, r\} = \{1, 2, 3\}$. Therefore, $\mathscr{C}$ has length at least $2(g - 4) + 4 > g$. To finalize, consider the case $w_1, w_2 \in \Omega \setminus \{\gamma_1, \gamma_2\}$ and $\{\gamma_1\gamma_2', \gamma_2\gamma_1', w_1w_1', w_2w_2'\} \subset E(\mathscr{C})$. Clearly, the cycle $\mathscr{C}$ must contain either (1) a (γ_1, γ_2)-path [or a (γ_1', γ_2')-path], or (2), a (w_1, γ_1)-path, and a (w_2, γ_2)-path. If (1), then $\mathscr{C}$ has length at least $d_{\tilde{C}_1}(\gamma_1, \gamma_2) + 4 \geqslant (g - 2) + 4 > g$. If (2), then $\mathscr{C}$ must also contain a (w_1', γ_1')-path and a (w_2', γ_2')-path, and thus its length is at least $4 + d_{\tilde{C}_1}(w_1, \gamma_1) + d_{\tilde{C}_1}(w_2, \gamma_2) + d_{\tilde{C}_1'}(w_1', \gamma_1') + d_{\tilde{C}_1'}(w_2', \gamma_2') = 4 + 2(d_{\tilde{C}_1}(w_1, \gamma_1) + d_{\tilde{C}_1}(w_2, \gamma_2)) \geqslant 4 + 2(g - 6) = 2g - 8 \geqslant g$, since $g \geqslant 8$.

Therefore, $L \geqslant 1$ for every nonsuperconnected $(3, g)$-cage with $g \geqslant 8$. $\square$

Lemma 2.3. *If G is a nonsuperconnected $(3, g)$-cage, then $L \leqslant \ell - 3$, where $\ell = \lfloor (g - 1)/2 \rfloor$.*

Proof. Suppose that G is a nonsuperconnected $(3, g)$-cage and $L \geqslant \ell - 2$. Without loss of generality, we can assume that $d_{C_1}(X, Y) = L$. Consider the subgraph $\tilde{C}_1 = G[V(C_1) \cup \{x, y\}]$, which is clearly connected. It is also clear that $d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g - 2$ (as in Lemma 2.2), $d_{\tilde{C}_1}(x, y) = L + 2 \geqslant \ell$, and by the minimality of L, $d_{\tilde{C}_1}(\gamma_i, x) \geqslant L + 1 \geqslant \ell - 1$ and $d_{\tilde{C}_1}(\gamma_i, y) \geqslant L + 1 \geqslant \ell - 1$, $i = 1, 2$. Next, take the subset $\Omega = \{x, y, \gamma_1, \gamma_2\}$ of $V(\tilde{C}_1)$, which has cardinality four. Notice that for all $v \in V(\tilde{C}_1) \setminus \Omega$, $\deg_{\tilde{C}_1}(v) = 3$, and for all $w \in \Omega$, $\deg_{\tilde{C}_1}(w) = 2$ (see Fig. 3).

Now, we construct the graph G^* in the same way as in Lemma 2.2. Note that $|V(G^*)| = 2|V(C_1)| + 4 < 2|V(C_1)| + 6 \leqslant |V(G)|$ (see Lemma 2.1), and $E^+ = \{w\varphi(w) : w \in \Omega\} = \{xx', yy', \gamma_1\gamma_2', \gamma_2\gamma_1'\}$. Let $\mathscr{C}$ be a cycle of G^*.

Suppose $|E(\mathscr{C}) \cap E^+| = 2$. Assume first that $\gamma_1\gamma_2'$ and $\gamma_2\gamma_1'$ are in $\mathscr{C}$. Since $d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g-2$, the cycle $\mathscr{C}$ has length at least $g-2+2=g$. Secondly, if xx' and yy' are in $\mathscr{C}$, then the length of $\mathscr{C}$ is at least $d_{\tilde{C}_1}(x, y) + d_{\tilde{C}_1'}(x', y') + 2 \geqslant 2L + 6 \geqslant 2\ell + 2 \geqslant g$. Finally, if xx' [or yy'] and $\gamma_1\gamma_2'$ [or $\gamma_2\gamma_1'$] are in $\mathscr{C}$, then $\mathscr{C}$ has length at least $d_{\tilde{C}_1}(x, \gamma_1) + d_{\tilde{C}_1'}(x', \gamma_2') + 2 = d_{\tilde{C}_1}(x, \gamma_1) + d_{\tilde{C}_1}(x, \gamma_2) + 2 \geqslant d_{\tilde{C}_1}(\gamma_1, \gamma_2) + 2 \geqslant g - 2 + 2 = g$.

Suppose $|E(\mathscr{C}) \cap E^+| = 4$, i.e., $\{xx', yy', \gamma_1\gamma_2', \gamma_2\gamma_1'\} \subset E(\mathscr{C})$. Then, $\mathscr{C}$ must contain either (1) a (γ_1, γ_2)-path, or (2) an (x, γ_i)-path and a (y, γ_{i+1})-path, where the sum for indexes is taken modulo 2. If (1), then $\mathscr{C}$ has length at least $d_{\tilde{C}_1}(\gamma_1, \gamma_2) + 4 \geqslant (g-2) + 4 > g$. If (2), then the length of $\mathscr{C}$ is at least $d_{\tilde{C}_1}(x, \gamma_i) + d_{\tilde{C}_1}(y, \gamma_{i+1}) + 4 \geqslant 2(\ell - 1) + 4 = 2\ell + 2 \geqslant g$.

Summarizing, we have proved that G^* is a 3-regular connected graph satisfying $g(G^*) \geqslant g(G) = g$ and $|V(G^*)| < |V(G)|$, which, according to Theorem 1.2, is a contradiction since G is a $(3, g)$-cage. Hence, $L \leqslant \ell - 3$. $\square$

At this point, we are ready to approach the proof of the main results of this work, which are displayed in one proposition and one theorem.

Proposition 2.1. *Every $(3, g)$-cage with $g \geqslant 10$ is superconnected.*

Proof. Let G be a nonsuperconnected $(3, g)$-cage with $g \geqslant 10$. Without loss of generality, assume $d_{C_1}(X, Y) = d_{C_1}(\alpha_1, \beta_1) = L$. By Lemmas 2.2 and 2.3, we have that $1 \leqslant L \leqslant \ell - 3$, where $\ell = \lfloor (g-1)/2 \rfloor$. Let Π denote the only path of length L in C_1 joining α_1 to β_1 and consider the set $F = N_{C_1}(V(\Pi)) \setminus V(\Pi)$, which has $|F| = |V(\Pi)| = L+1$. (See Fig. 6).

Note that $\{\alpha_2, \beta_2\} \cap F = \emptyset$, because $d_{C_1}(\alpha_2, F) \geqslant g - (L+3) \geqslant g - \ell \geqslant \ell + 1$, and $d_{C_1}(\beta_2, F) \geqslant g - (L+3) \geqslant \ell + 1$; and note also that $\{\alpha_2, \beta_2\} \cap \{\gamma_1, \gamma_2\} = \emptyset$ because $L \geqslant 1$. Now assume that $\gamma_1 \in F$ as illustrated in Fig. 6. The subgraph of G induced by $H = V(\Pi) \cup \{x, y, z, \gamma_1\}$ is a separating set of diameter at most $L + 3 \leqslant \ell$. From Theorem 1.3, it follows that $\ell \geqslant L + 3 \geqslant \lfloor g/2 \rfloor \geqslant \ell$. That is, the diameter of $G[H]$ is equal to $L + 3 = \ell$, but in this case $d_{G[H]}(u, v)$ is maximized for exactly one pair of vertices, so $\ell = L + 3 > \lfloor g/2 \rfloor \geqslant \ell$, a contradiction. Therefore, $F \cap \{\gamma_1, \gamma_2\} = \emptyset$ and the set $\Omega = F \cup \{\alpha_2, \beta_2\} \cup \{\gamma_1, \gamma_2\}$ has cardinality $4 + |F| = L + 5$.

Next, consider the subgraph $\tilde{C}_1 = C_1 - \Pi$, and notice that $\deg_{\tilde{C}_1}(v) = 3$ for all $v \in V(\tilde{C}_1) \setminus \Omega$, and $\deg_{\tilde{C}_1}(w) = 2$ for all $w \in \Omega$. In addition, we have that:

- $d_{\tilde{C}_1}(\gamma_1, \gamma_2) \geqslant g - 2$.
- $d_{\tilde{C}_1}(w_1, w_2) \geqslant g - L - 4$, for any pair of different vertices $w_1, w_2 \in F \cup \{\alpha_2, \beta_2\}$.
- $d_{\tilde{C}_1}(w_1, \gamma_1) + d_{\tilde{C}_1}(w_2, \gamma_2) \geqslant g - L - 5$, for any pair of different vertices $w_1, w_2 \in F \cup \{\alpha_2, \beta_2\}$, $\{w_1, w_2\} \neq \{\alpha_2, \beta_2\}$, since the union of any (w_1, γ_1)-path in $\tilde{C}_1$, any (w_2, γ_2)-path in $\tilde{C}_1$, the (w_1, w_2)-path in G (of length at most $L + 3$) and the (γ_1, γ_2)-path in G contains a cycle in G.
- $d_{\tilde{C}_1}(\alpha_2, \gamma_i) + d_{\tilde{C}_1}(\beta_2, \gamma_{i+1}) \geqslant \max\{2L, g - L - 6\}$, since $d_{\tilde{C}_1}(\alpha_2, \gamma_i) \geqslant L$ and $d_{\tilde{C}_1}(\beta_2, \gamma_{i+1}) \geqslant L$; and since the union of any (α_2, γ_i)-path in $\tilde{C}_1$, any (β_2, γ_{i+1})-path

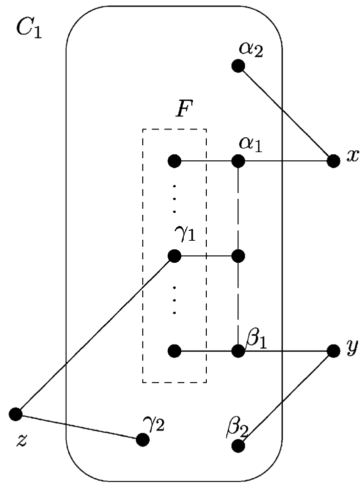

Fig. 6. Detail of a nonsuperconnected $(3, g)$-cage when $\gamma_1 \in F$.

in $\tilde{C}_1$, the (α_2, β_2)-path in G (of length $L + 4$) and the (γ_1, γ_2)-path in G contains a cycle in G.

Now, we construct the graph G^* in the same way as in Lemmas 2.2 and 2.3. Observe that $|V(G^*)| < |V(G)|$, $E^+ = \{w\varphi(w) : w \in \Omega\} = \{\alpha_2\alpha_2', \beta_2\beta_2', \gamma_1\gamma_2', \gamma_2\gamma_1'\} \cup \{uu' : u \in F\}$. Let $\mathscr{C}$ be a cycle of G^*.

Suppose $|E(\mathscr{C}) \cap E^+| = 2$. If $\gamma_1\gamma_2'$ and $\gamma_2\gamma_1'$ are in $\mathscr{C}$, then reasoning as in the above lemmas, $\mathscr{C}$ has length at least $g - 2 + 2 = g$. If $w_1, w_2 \in F \cup \{\alpha_2, \beta_2\}$ and $\{w_1w_1'w_2w_2'\} \subset E(\mathscr{C})$, then the length of $\mathscr{C}$ is at least $d_{\tilde{C}_1}(w_1, w_2) + d_{\tilde{C}_1'}(w_1', w_2') + 2 \geqslant 2(g - L - 4) + 2 \geqslant 2(g - \ell) > g$. If $w \in F \cup \{\alpha_2, \beta_2\}$ and $\{\gamma_i\gamma_{i+1}', ww'\} \subset E(\mathscr{C})$ for some $i = 1, 2$, then the cycle $\mathscr{C}$ has length at least $d_{\tilde{C}_1}(w, \gamma_i) + d_{\tilde{C}_1'}(w', \gamma_{i+1}') + 2 = d_{\tilde{C}_1}(w, \gamma_i) + d_{\tilde{C}_1}(w, \gamma_{i+1}) + 2 \geqslant d_{\tilde{C}_1}(\gamma_i, \gamma_{i+1}) + 2 \geqslant (g - 2) + 2 = g$.

Suppose $|E(\mathscr{C}) \cap E^+| = 4$. If $\mathscr{C}$ contains at least three edges w_iw_i', $w_i \in F \cup \{\alpha_2, \beta_2\}$, $i = 1, 2, 3$, then $\mathscr{C}$ must contain an (w_j, w_q)-path in $\tilde{C}_1$ and a (w_q', w_r')-path in $\tilde{C}_1'$, $\{j, q, r\} = \{1, 2, 3\}$. Therefore, $\mathscr{C}$ has length at least $2(g - L - 4) + 4 \geqslant 2(g - \ell) + 2 > g$. If $w_1, w_2 \in F \cup \{\alpha_2, \beta_2\}$ and $\{\gamma_1\gamma_2', \gamma_2\gamma_1', w_1w_1', w_2w_2'\} \subset E(\mathscr{C})$, then the cycle $\mathscr{C}$ must contain either (1) a (γ_1, γ_2)-path in $\tilde{C}_1$ [or an (γ_1', γ_2')-path in $\tilde{C}_1'$], or (2) a (γ_i, w_1)-path in $\tilde{C}_1$ and a (γ_{i+1}, w_2)-path in $\tilde{C}_1$. If (1), then $\mathscr{C}$ has length at least $4 + d_{\tilde{C}_1}(\gamma_1, \gamma_2) > g$. If (2), then $\mathscr{C}$ must contain also a (γ_i', w_1')-path in $\tilde{C}_1$ and a (γ_{i+1}', w_2')-path in $\tilde{C}_1$. Thus, $\mathscr{C}$ has length at least

$$
\begin{aligned}
&4 + 2d_{\tilde{C}_1}(w_1, \gamma_i) + 2d_{\tilde{C}_1}(w_2, \gamma_{i+1}) \\
&\quad \geqslant \begin{cases} 4 + 2(g - L - 5) \geqslant 2(g - \ell) > g & \text{if } \{w_1, w_2\} \neq \{\alpha_2, \beta_2\}, \\ 4 + 2\max\{2L, g - L - 6\} & \text{if } \{w_1, w_2\} = \{\alpha_2, \beta_2\}. \end{cases}
\end{aligned}
$$

Since $g \geqslant 10$, we have

$$4 + 2\max\{2L, g - L - 6\} = \begin{cases} 4 + 4L \geqslant g & \text{if } L \geqslant \lceil \frac{g-6}{3} \rceil, \\ 4 + 2(g - L - 6) \geqslant g & \text{if } L < \lceil \frac{g-6}{3} \rceil. \end{cases}$$

Suppose $|E(\mathscr{C}) \cap E^+| \geqslant 6$. In this case $\mathscr{C}$ contains at least four edges $w_i w_i'$, $w_i \in F \cup \{\alpha_2, \beta_2\}$, $i = 1, 2, 3, 4$, hence $\mathscr{C}$ has length at least g.

So, G^* is a 3-regular connected graph with fewer vertices than G and satisfies $g(G^*) \geqslant g(G) = g$, which is a contradiction if we take into account Theorem 1.2. □

Theorem 2.1. *Every* $(3, g)$*-cage* G *with* $g \geqslant 5$ *is quasi* 4*-connected.*

Proof. As it has already seen that the Petersen graph is quasi 4-connected (see Fig. 1), we may suppose G to have a girth $g \geqslant 6$. Let us begin by showing that G is superconnected. To this end, we distinguish the following cases (see [8,10] for the description of the first cubic cages):

- $g = 6$ (Heawood graph): In this case, $\ell - 3 = \lfloor (g-1)/2 \rfloor - 3 = -1$. In consequence, according to Lemma 2.3, G must be superconnected.
- $g = 7$ (McGee graph): Since its diameter is 4, this cubic cage is superconnected according to Theorem 1.1.
- $g = 8$ (Tutte's 8-cage): In this case, $\ell - 3 = 0$. If G was not superconnected, we would have $L = 0$ by Lemma 2.3, yielding a contradiction with Lemma 2.2. Then, G must be superconnected. Notice that this result can be also obtained from the fact that the diameter of G is equal to 4, taking into account Theorem 1.1.
- $g = 9$ (Brinkmann, McKay, Saager graphs): Every $(3, 9)$-cage is superconnected according to Theorem 1.1, since its diameter G is equal to 6 (see [2]).
- $g \geqslant 10$: See Proposition 2.1.

To end the proof, suppose G to be superconnected but not quasi 4-connected. This means that there exists a vertex v such that $N(v)$ is a cutset and $G - N(v)$ has three components. This fact allows us to derive that $\{v\} \cup N(v)$ is a cutset such that its induced subgraph $G[\{v\} \cup N(v)]$ has diameter 2. So, from Theorem 1.3, we get the contradiction $2 \geqslant \lfloor g/2 \rfloor \geqslant 3$. In consequence, every $(3, g)$-cage must be quasi 4-connected. □

3. Open questions

Certainly, a lot work must be done before approaching the proof of the *conjecture* of Fu, Huang and Rodger. A suitable way of doing it might be by firstly proving the following statements.

I. Every (δ, g)-cage, $\delta \geqslant 4$, is 4-connected.
II. Every $(4, g)$-cage is quasi 5-connected.
III. Every (δ, g)-cage is δ-connected for some particular range of g.

Conjecture. Every (δ, g)-cage is quasi $(\delta+1)$-connected.

Acknowledgements

The authors wish to thank the referees for their helpful comments and suggestions. This work was supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología, CICYT) under project TIC2000-1017.

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